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1993 J. Phys. A: Math. Gen. 26 L357

(http://iopscience.iop.org/0305-4470/26/7/001)

J. Phys. A: Math. Gen. 26 (1993) L357–L359. Printed in the UK

LETTER TO THE EDITOR

Linear covariance algebra for $SL_q(2)$

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Received 5 January 1993

Abstract. The Askey–Wilson algebra AW(3) is shown to serve as a 'hidden' covariance algebra for quantum algebra $SL_q(2)$. The generators of AW(3) are chosen to be linear combinations of $SL_q(2)$ generators with operator-valued coefficients.

The now widely investigated $SL_q(2)$ algebra [1] is not invariant under linear transformations of its generators and thus loses one of the most important and useful properties of the Lie algebras. Here we will show that this property may be restored by admitting the coefficients in linear combinations to be operator-valued factors.

The $SL_q(2)$ algebra is defined by commutation relations [1], [2]

$$[A_0, A_\pm] = \pm A_\pm \qquad [A_-, A_+] = u\exp(2\omega A_0) + v\exp(-2\omega A_0) \tag{1}$$

where u and v are real parameters, $\omega > 0$. The special cases of the algebra (1) and $SU_q(2)$ $(u = -v < 0)$, $SU_q(1,1)$ $(u = -v > 0)$, $CU_q(2)$ $(u = v > 0)$ and $EU_q(2)$ $(uv = 0)$ [2] ($CU_q(2)$ and $EU_q(2)$ are known as q-oscillator algebras [2–4]).

The Casimir operator of $SL_q(2)$ commuting with its generators is

$$\hat{Q} = A_+A_- + \frac{v\exp(-2\omega A_0) - u\exp(2\omega(A_0 - 1))}{1 - \exp(-2\omega)}. \tag{2}$$

In what follows we will proceed to the space with fixed value Q of the Casimir operator $\hat{Q}$, i.e. this means that the expressions A_+A_-, A_-A_+, etc, will be reduced by means of equation (2).

Consider the operators

$$\begin{aligned} K_1 &= \alpha_1\exp(\omega A_0)A_+ + \alpha_2 A_-\exp(\omega A_0) + \alpha_3\exp(2\omega A_0) \\ K_2 &= \beta_1\exp(-\omega A_0)A_+ + \beta_2 A_-\exp(-\omega A_0) + \beta_3\exp(-2\omega A_0) \end{aligned} \tag{3}$$

where α_i, β_i are arbitrary complex parameters.

Then it can be directly verified that these operators together with their q-mutator

$$K_3 = [K_1, K_2]_\omega \equiv e^{\omega}K_1K_2 - e^{-\omega}K_2K_1 \tag{4}$$

are closed in frames of linear algebra under q-mutations

$$\begin{aligned} [K_2, K_3]_\omega &= C_1K_1 + BK_2 + D_1 \\ [K_3, K_1]_\omega &= C_2K_2 + BK_1 + D_2. \end{aligned} \tag{5}$$

Explicit formulas for the structure constants B, C_i, D_i are given in the appendix.

The algebra (5) is known as Askey–Wilson (AW(3)) algebra [5]. Due to simple q-mutation relations its representations, spectra, overlaps and other properties are immediately obtained (for details see [5]).

For example, if the spectrum of K_1 is a purely discrete one and the spectrum of K_2 is mixed (this happens for discrete positive series of $SU_q(1,1)$ if $\alpha_1=\alpha_2^*$, $\beta_1=\beta_2^*$) then the overlaps between eigenstates of these operators are expressed in terms of Askey-Wilson polynomials, introduced in [6]. (Relations between these polynomials and representations of AW(3) algebra are considered in [5].) Analogously to the case of Lie algebra these overlaps have the meaning of the generalized spherical functions for $SL_q(2)$ algebra. (In [7], a similar approach to the construction of spherical functions for $SU_q(2)$ was proposed, however covariance property (3)-(5) was not found.)

Thus, AW(3) algebra serves as 'hidden' covariance algebra being a natural extension of linear covariance property of ordinary Lie algebras.

The covariance property (3)-(5) allows one to obtain many useful results for $SL_q(2)$ itself. For example, choosing

$$\alpha_3=\beta_3=0 \qquad \alpha_1\alpha_2=-e^{-\omega}/4\cosh\omega\sinh\omega=e^{\omega}/4\cosh\omega\sinh 2\omega$$

we obtain the 'Cartesian' version of $SL_q(2)$

$$[K_a, K_b]_\omega=e_{abc}K_c. \tag{6}$$

(e_{abc} is a completely antisymmetric tensor). This form is an exact q-analogue of ordinary rotation algebra $O(3)$; the 'Cartesian' version for $SU_q(2)$ was considered in [8].

Great flexibility of transformations (3) gives rise to much improvement and simplifications in quantum algebra theory. Indeed, one can show that the same AW(3) algebra plays the role of 'hidden' symmetry for Clebsch-Gordan and Racah problems for $SL_q(2)$ [9, 10]. Moreover, there is a remarkable 'hidden' relation between Racah, Clebach-Gordan and Wigner problems not having any 'classical' analogue [9].

Appendix

In this appendix we present explicit formulae for the structure constants of AW(3) algebra on given realization (3) with fixed value Q of the Casimir operator $\hat{Q}$ (2).

Let us write the first q-mutator in (5) in the form

$$[K_2, K_3]_\omega\equiv 2\cosh 2\omega K_2K_1K_2-(K_1K_2^2+K_2^2K_1). \tag{A1}$$

We see that this q-mutator is linear in K_1. So, in order to obtain the first q-mutation in (5) it is sufficient to evaluate it separately for $K_1=\exp(\omega A_0)A_+$, $A_-\exp(\omega A_0)$, $\exp(2\omega A_0)$ then sum the results. Thus we obtain

$$B=4\sinh^2\omega(\alpha_3\beta_3+Q\tau) \tag{A2}$$

$$C_1=4u\,e^{-\omega}\beta_1\beta_2\cosh\omega\sinh 2\omega \tag{A3}$$

$$D_1=2\sinh 2\omega(u\beta_3\tau+2Q\,e^{-\omega}\beta_1\beta_2\alpha_3\sinh\omega) \tag{A4}$$

where

$$\tau=\alpha_1\beta_2+\alpha_2\beta_1. \tag{A5}$$

Analogously, writing the second q-mutator in (5) in the form

$$[K_3, K_1]_\omega\equiv 2\cosh 2\omega K_1K_2K_1-(K_1^2K_2+K_2K_1^2) \tag{A6}$$

we obtain the remaining coefficients

$$C_2 = -4ve^{\omega}\alpha_1\alpha_2 \cosh \omega \sinh 2\omega$$

$$D_2 = 2 \sinh 2\omega(2Qe^{\omega} \sinh \omega\alpha_1\alpha_2\beta_3 - v\alpha_3\tau). \tag{A7}$$

In contrast to the case of Lie algebras, the structure constants B, D_1, D_2 contain Casimir value Q, i.e. the form of AW(3) algebra (5) depends on representation of $SL_q(2)$.

Note obvious symmetry of the coefficients $C_1 \leftrightarrow C_2$, $D_1 \leftrightarrow D_2$, $B \leftrightarrow B$ under the exchange $u \leftrightarrow v$, $\omega \to -\omega$, $\alpha_i \leftrightarrow \beta_i$.

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